# Combinatorial Concepts With Sudoku
# I: Symmetry

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**Abstract**

The immensely popular logic game Sudoku incorporates a significant number of combinatorial concepts. In this essay, I use the mathematical and recreational study of Sudoku as a motivating example to introduce these concepts. Note that I *do not* consider techniques or strategies for *playing* Sudoku as these are extensively covered elsewhere.

## 1 Symmetry in Sudoku

Many of the Sudoku puzzles published in newspapers and elsewhere have the property that the initial pattern of non-empty squares is *symmetric*.

Most people would intuitively agree that the pattern of clues in the puzzle in Figure 1 is highly symmetric, but what exactly does *symmetric* mean and how can we quantify the different types of symmetry that occur in Sudoku puzzles? The precise answer to these questions leads us to a branch of mathematics called *group theory*.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 2 | | | 3 | | | 4 | |
| 6 | | | | | | | | 3 |
| | | 4 | | | | 5 | | |
| | | | 8 | | 6 | | | |
| 8 | | | | 1 | | | | 6 |
| | | | 7 | | 5 | | | |
| | | 7 | | | | 6 | | |
| 4 | | | | | | | | 8 |
| | 3 | | | 4 | | | 2 | |

Figure 1: Symmetric Sudoku

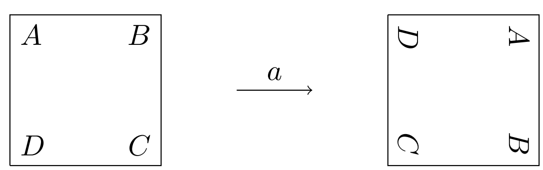
| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 4 | | | 8 | | | 6 | |
| 3 | | | | | | | | 2 |
| | | 7 | | | | 4 | | |
| | | | 7 | | 8 | | | |
| 4 | | | | 1 | | | | 3 |
| | | | 5 | | 6 | | | |
| | | 6 | | | | 5 | | |
| 2 | | | | | | | | 4 |
| | 8 | | | 6 | | | 3 | |

Figure 2: Rotated Symmetric Sudoku

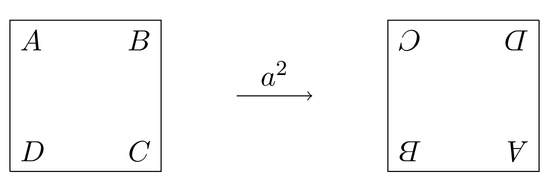
A shape or pattern is called *symmetric* if it can be transformed in some way without changing the way that it looks; for example we can *rotate* the puzzle of Figure 1 by 90° clockwise and, as shown in Figure 2, the pattern of clues looks the same.

So what are the possible transformations that can be symmetries of a Sudoku puzzle? Clearly the transformation must leave the overall square shape of the grid unchanged, so we can start by considering what operations do this — in technical language we are going to identify the *symmetries of a square.*

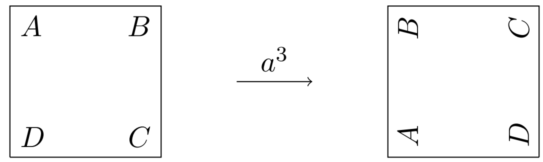
We have already seen that one of these symmetries is *rotation by* 90°—we'll give this operation a name and call it $a$.




Now clearly we can *repeat* this operation, rotating by another 90°, thereby getting a 180° rotation in total. As this operation arises by performing $a$ twice, we'll call it $aa$ or just $a^2$.

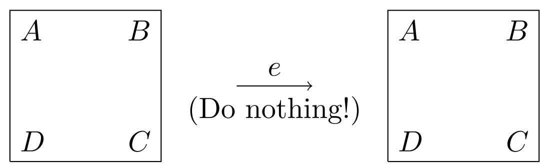



We can repeat this once more, getting a 270° rotation that, unsurprisingly, we will call $a^3$.



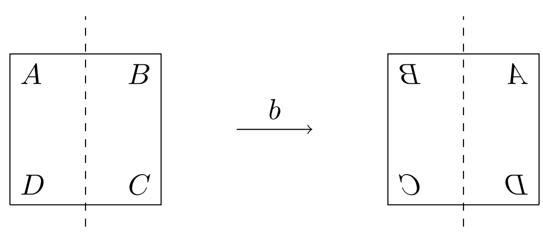

What happens if we do *another* 90° rotation? This corresponds to doing a full 360° rotation, which is the same as doing nothing at all. Should we have a name for the operation "doing nothing"? Although it may seem superfluous, it turns out to be useful to give this operation a name, and so we'll give it the special name $e$, and we have the equation

$$a^4 = e.$$

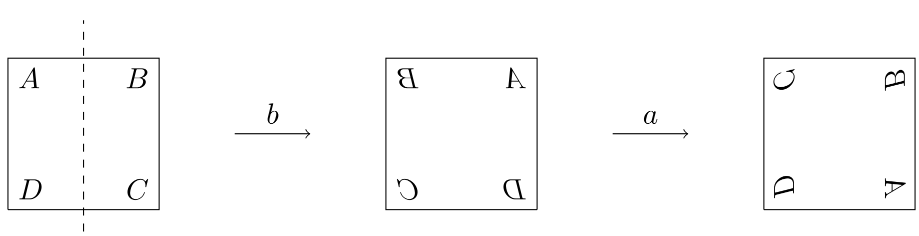



We say that $a$ has *order* 4 because applying it 4 times brings the square back to its original position *for the first time*. Obviously applying it 8 times, or 12 times, or 20 times will also bring the square back to its original position, but not for the first time. The 180° rotation $a^2$ has order 2 because we only need to apply that twice in order to return to the original position. You can check that $a^3$ has the same order as $a$.

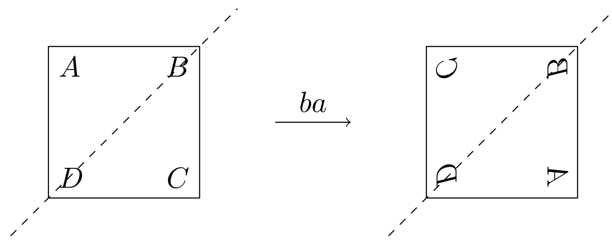
As well as the rotations, certain *reflections* are also symmetries of the square. For example, we can reflect the square through a vertical line, as shown below. We will call this reflection operation $b$.



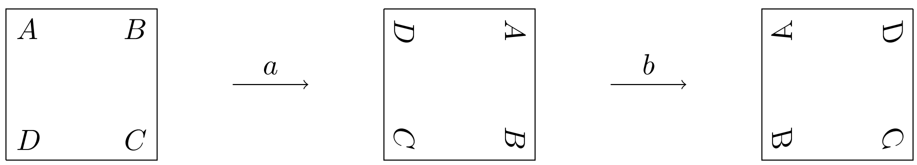

One property of symmetries that we used above is that if you *combine* two or more symmetries, then you get another one. For example, what happens if we *first* do a reflection $b$ and then the rotation $a$?



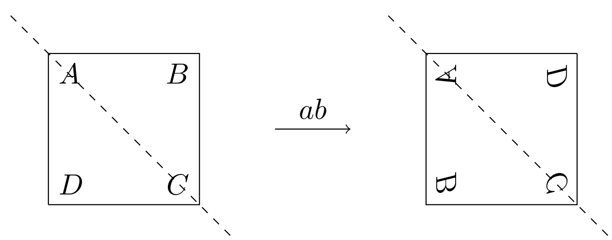
This combined operation, which is called $ba$, has the same effect as a *diagonal reflection* through the diagonal line running from the bottom-left to the top-right of the square; this is known as the *anti-diagonal*.



Notice that if we do $a$ first and *then* $b$, we do not get the same result!



In fact, $ab$ is equal to the *other* diagonal reflection, this time in the *main diagonal*.



It is easy to verify that this diagonal reflection can also be obtained by first doing $b$ and then $a^3$, so we get the equation

$$ab = ba^3.$$

Are there any more symmetries of the square? There is obviously at least one more missing, which is the reflection through a horizontal axis; it will come as no surprise to discover that this can be obtained as the combination $ba^2$.

So now, we have 8 symmetries of the square as follows:

| Name | Symmetry | Order |
|---|---|---|
| $e$ | Identity (do nothing) | 1 |
| $a$ | 90° clockwise rotation | 4 |
| $a^2$ | 180° rotation | 2 |
| $a^3$ | 270° clockwise rotation | 4 |
| $b$ | Reflection in vertical axis | 2 |
| $ba$ | Reflection in anti-diagonal | 2 |
| $ba^2$ | Reflection in horizontal axis | 2 |
| $ba^3$ | Reflection in main diagonal | 2 |

Now we can work out fairly quickly that no matter what combination of these symmetries we apply to the square, there is no way of getting a new one that is not on the list. For example, we might consider what happens if we first reflect in the horizontal axis ($ba^2$) and then in the main diagonal ($ba^3$). The combined operation is then

$$ba^2ba^3 = ba^2(ab) = ba^3b = (ab)b = ae = a$$

where we twice used the fact that $ba^3 = ab$ and then that $b^2 = e$.

In fact, it turns out that these are *all* of the symmetries of the square and that any combination of these symmetries is already in the list — this means that the collection of 8 symmetries listed above forms a *group*.

Definition

A *group* is a set $G$ together with a binary operation $\cdot$ and a special element $e \in G$ satisfying the following properties:

- [Closure] If $x$, $y \in G$ then $x \cdot y \in G$
- [Associativity] For all $x$, $y$, $z \in G$ we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
- [Identity] For all $x \in G$ we have $x \cdot e = e \cdot x = x$
- [Inverses] For each $x \in G$ there is an element $x^{-1} \in G$ such that $x \cdot x^{-1} = x^{-1} \cdot x = e$

Normally we simply use $xy$ to represent $x \cdot y$.

For the symmetries of a square, we have already been using the idea of combining two symmetries by performing one after the other, and this is the binary operation for this group. This group contains 8 elements and is called the *dihedral group* of order 8, usually denoted $D_8$ (though some authors denote it $D_4$ just to confuse everyone).

For this group of symmetries, the *inverse* of an element $x$ is essentially the transformation that "undoes" whatever $x$ did. For example the inverse of the 90° rotation $a$ is the 270° rotation $a^3$, because doing these two rotations one after another (in either order) is the same as doing nothing. In words we say that "the inverse of $a$ is the element $a^3$ or in symbols that

$$a^{-1} = a^3.$$

Now a Sudoku puzzle may happen to have all of these operations as symmetries (like the one in Figure 1) but it is also possible for a puzzle to have just *some* of them; for example Figure 3 shows a puzzle whose *only* non-identity symmetry is reflection in the main diagonal[1].We assume that *every* puzzle has the identity or "do-nothing" operation as a symmetry, and so the full list of symmetries for this puzzle is $\{e, ba^3\}$.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |
| | | | | | | | 1 | 2 |
| | | 3 | | 4 | 5 | | | |
| | | | 6 | | 1 | | 7 | |
| | | 4 | | | | 6 | | |
| | | 5 | 8 | | | | | |
| | | | | 3 | | 4 | | |
| | 1 | | 2 | | | | | |
| | 7 | | | | | | | |

Figure 3: A puzzle with $ba^3$ as its only non-identity symmetry

If puzzles such as the ones in Figure 4 and Figure 5 has the rotation $a$ as a symmetry, then they *must* also have $a^2$ and $a^3$ (and of course $e$) as symmetries, and so we cannot find a puzzle with $a$ as its *only* non-identity symmetry. Similarly if a puzzle has $x$ and $y$ as symmetries, then it must also have $xy$ as a symmetry. This means that the set of symmetries of any puzzle must *themselves* form a group, which is called a *subgroup* of $D_8$.

> DEFINITION
> A subset $H$ of a group $G$ is called a *subgroup* of $G$ if the elements of $H$ together with the binary operation $\cdot$ and identity element $e$ from $G$ form a group.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 8 | | 5 | | | | | |
| | 9 | 6 | 7 | | | | 2 | 4 |
| | | | | | | | 9 | |
| | | | | | | | 7 | 2 |
| | | | | | | | | |
| 3 | 1 | | | | | | | |
| | 4 | | | | | | | |
| 2 | 5 | | | | 3 | 8 | 1 | |
| | | | | | 8 | | 3 | |

Figure 4: A puzzle with symmetry group $\{e, a, a^2, a^3\}$

[1]Although I found this puzzle, it was first put into this form by "Red Ed" from the `www.sudoku.com` forums.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | 2 | | |
| | 8 | | | | 7 | | 9 | |
| 6 | | 2 | | | | 5 | | |
| | 7 | | | 6 | | | | |
| | | | 9 | | 1 | | | |
| | | | | 2 | | | 4 | |
| | | 5 | | | | 6 | | 3 |
| | 9 | | 4 | | | | 7 | |
| | | 6 | | | | | | |

Figure 5: Another puzzle with symmetry group $\{e, a, a^2, a^3\}$

The *order* of a group is the number of elements in the group, and it is a standard result in group theory that the order of a subgroup must be a divisor of the order of the group. Therefore seeing $D_8$ has order 8, it follows that any Sudoku puzzle must have exactly 1, 2, 4 or 8 symmetries. A subgroup of order 1 must consist only of the single element $\{e\}$ while a subgroup of order 2 must consist of $e$ together with one of the 5 possible elements of order 2 (namely $a^2$, $b$, $ba$, $ba^2$ or $ba^3$) and the subgroup of order 8 must be the entire dihedral group $D_8$.

We have already seen that it is possible to have a symmetry group of order 4 consisting only of rotations (Figure 4 and Figure 5). And if a subgroup contains either $a$ or $a^3$, then it must contain this entire subgroup. So, are there any other subgroups of order 4? Such a subgroup must contain $e$ and three other elements that cannot include $a$ or $a^3$; a quick calculation shows that it cannot contain three elements from $\{b, ba, ba^2, ba^3\}$ because if it did, then some combination of them would be equal to $a$. Therefore a subgroup of order 4 must contain $e$, $a^2$ and exactly two elements from the set $\{b, ba, ba^2, ba^3\}$. It is then easy to see that there are precisely two other possibilities for a group of order 4, which are

$$\{e, a^2, b, ba^2\} \qquad \text{and} \qquad \{e, a^2, ba, ba^3\}.$$

We can find Sudoku puzzles with exactly these symmetry groups, and these are shown in Figure 6 and Figure 7 (warning: this one is *hard* to solve).

Now, it is easy to see that if we have a puzzle with a reflection symmetry, for example $b$, then by transposing the matrix (that is, swapping rows and columns) we immediately get a puzzle with the reflection symmetry $ba^2$. So in some sense a horizontal reflection is the same "type" of symmetry as a vertical reflection. We can make this concept precise by using the group theoretic notion of *conjugacy*; two elements $x$ and $y$ in a group $G$ are *conjugate* if

$$y = z^{-1}xz$$

for some $z \in G$. For example, $ba^2$ is conjugate to $b$ because (using $a$ as the conjugating element $z$) we get

$$a^{-1}(ba^2)a = a^3ba^3 = baa^3 = b.$$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5 | | | | | | | | 4 |
| 9 | | | | | | | | 1 |
| | 8 | | 3 | | 4 | | 9 | |
| | 6 | | | | | | 4 | |
| | | | 2 | | 6 | | | |
| | 9 | | | | | | 3 | |
| | 7 | | 9 | | 5 | | 6 | |
| 2 | | | | | | | | 5 |
| 1 | | | | | | | | 9 |

Figure 6: A puzzle with symmetry group $\{e, b, a^2, ba^2\}$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6 | | | | | | | | 3 |
| | 7 | | | 8 | | | 9 | |
| | | 2 | | | | 5 | | |
| | | | 3 | | | | | |
| | 8 | | | 1 | | | 7 | |
| | | | | | 2 | | | |
| | | 5 | | | | 1 | | |
| | 9 | | | 4 | | | 8 | |
| 3 | | | | | | | | 2 |

Figure 7: A puzzle with symmetry group $\{e, ba, a^2, ba^3\}$

Therefore, treating conjugate subgroups of $D_8$ as representing the same type of symmetry, we end up with the following 7 distinct possibilities for a symmetric Sudoku puzzle:

| Type | Order | Description | Group |
|---|---|---|---|
| Type I | 8 | Full dihedral symmetry | $D_8$ |
| Type II | 4 | Full rotational symmetry | $\{e, a, a^2, a^3\}$ |
| Type III | 4 | Horizontal and vertical reflection | $\{e, b, a^2, ba^2\}$ |
| Type IV | 4 | Diagonal and anti-diagonal reflection | $\{e, ba, a^2, ba^3\}$ |
| Type V | 2 | 180° rotational symmetry | $\{e, a^2\}$ |
| Type VI | 2 | Horizontal *or* vertical reflection | $\{e, b\}$ or $\{e, ba^2\}$ |
| Type VII | 2 | Diagonal *or* anti-diagonal reflection | $\{e, ba\}$ or $\{e, ba^3\}$ |

## Version History

Version 0.2: 2006-03-29: Updated based on comments from Adam Glesser and typos spotted by Glenn Fowler.
Version 0.1: 2006-03-28: Originally Posted to `www.sudoku.com`.